OPTIMAL HEATING OF THREE-LAYER CYLINDRIC SHELLS WITH A LIGHT ELASTIC FILLER

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The problem is analyzed of determining the extremal temperature fields in three-layer cylindric shells ensuring a relatively low level of temperature stress. It is shown that the optimal temperature fields, as well as the arising temperature stresses, depend strongly on mechanical characteristics of a shell. Solutions for this class of problems for single-layer isotropic shells considered before within the framework of the classical Kirchhoff–Love theory are given in [1, 2].

1. Initial Equations

Let an infinite three-layer cylindric shell be subjected to an axially symmetric constant, relative to the thickness, temperature field T(x) (x is a parallel to the generatrix coordinate; R, h are shell radius and thickness; t, c is the thickness of carrier layers and of the filler, respectively). In this case if one regards the distribution of the tangential displacements of the carrier layers as constant the starting equations of the axially symmetric thermoelastic problem for three-layers shells with a light elastic filler susceptible only to transversal shift can be written as [3]

$$\frac{d^2\alpha}{dx^2} - \frac{2G_3}{cB}\left(\alpha - \frac{dw}{dx}\right) = 0; \tag{1.1}$$

$$C'\frac{d^3\alpha}{dx^3} + \frac{B(1-\mu^2)}{R}\left(\frac{w}{R} + \omega T\right) = 0,$$

where $B = Et/(1-\mu^2)$; $C' = Et(c+t)^2/2(1-\mu^2)$; E, μ is Young's modulus and Poisson's coefficient of the carrier layers; G_3 is the rigidity modulus of the filler; ω is the coefficient of temperature expansion; $\alpha = (u_1 - u_2)/(c+t)$; u_1, u_2 are displacements of the points of the mid-surface between the upper and the lower layers.

By eliminating the deflection w from Eqs. (1.1) one arrives at an equation for the rotation angle,

$$\frac{d^4\alpha}{d\xi^3} - 4\varkappa a^2\frac{d^2\alpha}{d\xi^2} + 4\alpha = -4a\omega R\frac{dT}{d\xi}, \tag{1.2}$$

where

$$\xi = ax;\ 4a^4 = B(1-\mu^2)/C'R^2;\ \varkappa = cB/2G_3.$$

If the function α is known the deflection can be obtained from the following expression:

$$\frac{dw}{d\xi} = \frac{\alpha}{a} - \varkappa a\frac{d^2\alpha}{d\xi^2},$$

and the overall annular force and the bending moment of a three-layer shell is given by

$$N = -2B(1-\mu^2)\left(\frac{w}{R} + \omega T\right);\ M = -C'a\frac{d\alpha}{d\xi}. \tag{1.3}$$

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2. Formulation of the Problem

The elastic energy of a three-layer shell is considered [3]:

$$\Pi = \frac{1}{2}\iint_{\Omega}\left\{\int_{-1/2c}^{1/2c}\tau_{xz}\gamma_{xz}dz + \int_{1/2c}^{1/2c+t}[\sigma_x(e_x-\omega T)+\sigma_y(e_y-\omega T)]\,dz + \int_{-1/2c-t}^{-1/2c}[\sigma_x(e_x-\omega T)+\sigma_y(e_y-\omega T)]\,dz\right\}d\Omega, \quad (2.1)$$

where Ω is the coordinate region corresponding to the mid-surface.

Integrating in (2.1) and using the distribution of displacements and stresses along the layer thickness, one obtains [3]

$$\Pi = \pi Ra\int_{-\infty}^{\infty}\left[G_3\frac{(c+t)^2}{c}\left(\alpha - a\frac{dw}{d\xi}\right)^2 + a^2C'\left(\frac{d\alpha}{d\xi}\right)^2 + B(1-\mu^2)\left(\frac{w}{R}+\omega T\right)^2\right]d\xi.$$

By eliminating the functions w and T with the aid of Eqs. (1.1) one can write the equation of the elastic three-layer shell in the following final form:

$$\Pi = \pi RC'a^3\int_{-\infty}^{\infty}\left[\frac{1}{4}\left(\frac{d^3\alpha}{d\xi^3}\right)^2 + a^2\varkappa\left(\frac{d^2\alpha}{d\xi^2}\right)^2 + \left(\frac{d\alpha}{d\xi}\right)^2\right]d\xi. \quad (2.2)$$

One should mention here that the elastic energy of a shell is represented by a positive-definite quadratic form of its arguments which vanish if and only if all the temperature stresses also vanish. It seems appropriate to adopt the stationarity condition for the elastic energy (2.2) of a shell as a criterion for finding optimal temperature fields ensuring a comparatively low level of temperature stresses.

The elastic energy (2.2) can be regarded as a functional defined on the set of functions α. The problem now arises of finding the functions $\alpha(x)$ for which the functional (2.2) attains its extremum and which satisfy Eq. (1.2), as well as the damping relations at infinity, and also some additional constraints on the function of deflections w, the rotation angles α, and the forces and moments. These conditions can be reduced to the following conditions for the functions at fixed sections of the shell, $\xi = \xi_j$ (j = 1, 2, ..., n):

$$a\frac{d^{(i)}\alpha(\xi_j)}{d\xi^i} = \alpha_{ij}; \quad a\int\left[\alpha\xi_j - \varkappa a^2\frac{d^2\alpha(\xi_j)}{d\xi^2}\right]d\xi = w_{ij}, \quad (i=0,1,2). \quad (2.3)$$

The formulated variational problem is equivalent to the following isoperimetric problem: to find an extremum of the functional $\Pi(\alpha)$ on the set of functions $\alpha(x)$ on which the functionals

$$\Pi_{ij}(\alpha) = (-1)^i a\int_{-\infty}^{\infty}\delta^{(i)}(\xi-\xi_j)\,\alpha(\xi)\,d\xi;$$

$$\Pi_j(\alpha) = a\int_{-\infty}^{\infty}\left[\alpha - \varkappa a^2\frac{d^2\alpha}{d\xi^2}\right]S_+(\xi_j-\xi)\,dx$$

assume the given values

$$\Pi_{ij}(\xi_j) = \alpha_{ij}; \quad \Pi_j(\xi_j) = w_j.$$

In the above $\delta^{(i)}(\xi) - i$ is the i-th derivative of the delta function; $S_+(\xi)$ is the discontinuity jump.

The above problem can be reduced to the finding of the absolute extremum of a functional [1, 4]:

$$\Pi_*(\alpha) = \frac{\pi RC'a}{4}\int_{-\infty}^{\infty}\left\{\left(\frac{d^3\alpha}{d\xi^3}\right)^2 + 4\varkappa a^2\left(\frac{d^2\alpha}{d\xi^2}\right)^2 + 4\left(\frac{d\alpha}{d\xi}\right)^2 - \right.$$
$$\left. - 2a'\left[\alpha\sum_{j=1}^{n}\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(\xi-\xi_j) + \left(\alpha - \varkappa a^2\frac{d^2\alpha}{d\xi^2}\right)\sum_{j=1}^{n}\lambda_j S_+(\xi_j-\xi)\right]\right\}d\xi, \quad (2.4)$$

where $a' = aR$; λ_{ij}, λ_j are arbitrary constants to be determined.

By writing down the Euler equation

$$\frac{d^6\alpha}{d\xi^6} - 4\varkappa a^2\frac{d^4\alpha}{d\xi^4} + 4\frac{d^2\alpha}{d\xi^2} = -a'\sum_{j=1}^{n}\left\{\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(\xi-\xi_j) + [\lambda_j S_+(\xi_j-\xi) - \varkappa a^2\lambda_j\delta'(\xi-\xi_j)]\right\}, \quad (2.5)$$

for the functional (2.4) one obtains a complete system of equations consisting of (2.5) together with Eq. (1.2) and the conditions (2.3) for finding the functions $\alpha(\xi)$, $T(\xi)$; consequently, one also finds the annular forces, bending moment, as well as the Lagrange multipliers which are of interest to us.

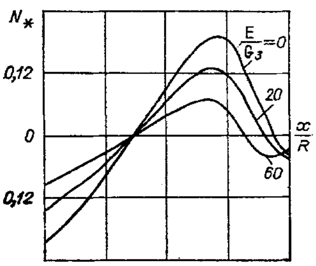

Fig. 1

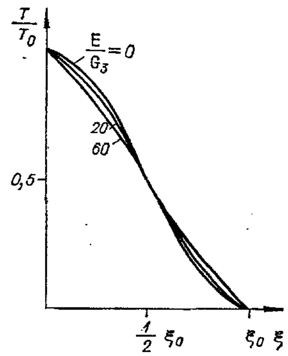

Fig. 2

3. Solving the Extremal Problem

By virtue of (1.2) one obtains from (2.6) the following equation for the extremal temperature field:

$$\frac{d^3T}{d\xi^3} = \frac{1}{4\omega}\sum_{j=1}^{n}\left\{\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(\xi-\xi_j) + \lambda_j[S_+(\xi_j-\xi) - \varkappa a^2\delta'(\xi-\xi_j)]\right\}. \tag{3.1}$$

The solutions of Eqs. (1.2) and (3.1) which vanish at infinity are found by applying the Fourier transformation. Then for the temperature one finds the expression

$$T(\xi) = \frac{1}{8\omega}\sum_{j=1}^{n}\left\{\lambda_j\frac{(\xi-\xi_j)^3}{6} - \varkappa a^2(\xi-\xi_j) + \lambda_{0j}\frac{(\xi-\xi_j)^2}{2} + \lambda_{1j}(\xi-\xi_j) + \lambda_{2j}\right\}\operatorname{sgn}(\xi-\xi_j).$$

Depending on the roots of the characteristic equation,

$$k^4 + 4\varkappa a^2 k^2 + 4 = 0$$

the solution for the functions $\alpha(\xi)$ may assume diverse forms. In the case of all complex roots

$$k_{1,2,3,4} = \pm\sqrt{-2\varkappa a^2 \pm \sqrt{(2\varkappa a^2)^2 - 4}} = \pm(r \pm is)$$
$$\left(r = \sqrt{1-\varkappa a^2};\quad s = \sqrt{1+\varkappa a^2}\right)$$

the solution is as follows:

$$\alpha(\xi) = -\frac{a'}{8}\sum_{j=1}^{n}\left\{\left[\frac{\lambda_j}{2}(\xi-\xi_j)^2 - 2\varkappa a^2) + \lambda_{0j}(\xi-\xi_j) + \right.\right.$$
$$\left. + \lambda_{1j}\right]\operatorname{sgn}(\xi-\xi_j) + \frac{e^{-s|\xi-\xi_j|}}{rs}\left[\frac{\lambda_j}{2}(2\varkappa a^2 rs\cos r(\xi-\xi_j) + \right.$$
$$+ (1-2\varkappa^2a^4)\sin r|\xi-\xi_j|) + \frac{\lambda_{0j}}{2}((1+2\varkappa a^2)r\cos r(\xi-\xi_j) -$$
$$-(1-2\varkappa a^2)s\sin r|\xi-\xi_j|) - \lambda_{1j}(rs\cos r(\xi-\xi_j) +$$
$$\left.\left. + \varkappa a^2\sin r|\xi-\xi_j|)\operatorname{sgn}(\xi-\xi_j) + \lambda_{2j}(r\cos r(\xi-\xi_j) + s\sin r|\xi-\xi_j|)\right]\right\}.$$

If the functions $T(\xi)$ and $\alpha(\xi)$ are known one can obtain the forces and moments in the shell by employing (1.3). Then the conditions at infinity and the continuity conditions determine the constraints imposed on the Lagrange coefficients:

$$\sum_{j=1}^{n}[\lambda_j(\xi_j^3 - 6\varkappa a^2\xi_j) - 3\lambda_{0j}\xi_j^2 - 6\lambda_{1j}\xi_j - 6\lambda_{2j}] = 0;$$
$$\sum_{j=1}^{n}[\lambda_j(\xi_j^2 - 2\varkappa a^2) - 2\lambda_{0j}\xi_j + 2\lambda_{1j}] = 0;$$
$$\sum_{j=1}^{n}(\lambda_j\xi_j - \lambda_{0j}) = 0;\quad \sum_{j=1}^{n}\lambda_j = 0.$$

4. Local Heating of Three-Layer Cylindric Shell

As an example local heating is considered of a three-layer cylindric shell with a light elastic filter. Let the temperature of the heating reach its maximal value T_0 at the section $\xi = 0$ and let it vanish at the sections $\xi = \xi_0$. Then the family of extremal temperature fields which are symmetric with respect to the section $\xi = 0$ is as follows:

$$T(\xi) = \frac{T_0}{1 + 12\frac{\varkappa a^2}{\xi_0^2}}\left\{2\left|\frac{\xi}{\xi_0}\right|^3 - 3\left(\frac{\xi}{\xi_0}\right)^2 + 1 + 12\frac{\varkappa a^2}{\xi_0^2}(|\xi| - \xi_0)\right\} \tag{4.1}$$

$$\text{for} \quad |\xi| \leqslant \xi_0; \; T(\xi) = 0 \quad \text{for} \quad |\xi| \geqslant \xi_0.$$

It can be seen that the distribution (4.1) depends on the ratio of Young's modulus of the carrier layers E to the rigidity modulus G_3 of the filler, and also on the ratio t/c of the thicknesses of the layers, and finally on the relative thickness h/R of the packet.

In the limit for $E/G_3 \to 0$ (the filler is absolutely rigid to shear) one has

$$\lim_{\varkappa \to 0} T(\xi) = T_0\left[2\left|\frac{\xi}{\xi_0}\right|^3 - 3\left(\frac{\xi}{\xi_0}\right)^2 + 1\right],$$

which is identical with the corresponding result [1] in the theory of thin isotropic shells obtained by using the classical Kirchhoff–Love theory.

In Fig. 1 the profiles are shown of the optimal temperature fields $T_* = T/T_0$ for a three-layer cylindrical shell with the following characteristics:

$$h/R = 1/20; \; t/c = 1/25; \; \mu = 0.3$$

versus the rigidity ratio E/G_3. The case $E/G_3 = 0$ corresponds to the solution [1].

In Fig. 2 graphs are shown of dimensionless quantities of the annular forces $N_* = N/B(1-\mu^2)\omega T_0$ calculated by employing the linear temperature distribution (4.1). It can be seen from these calculations that with the ratio E/G_3 increasing the profiles of the temperature fields are only slightly modified but the computational effort is considerably reduced.

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